Performance Analysis of Type-II Hybrid ARQ Systems

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***Abstract*—In this paper, an accurate performance analysis is proposed for type-II hybrid ARQ scheme. As opposed to the existing type-II hybrid ARQ analysis work in the literature, the analysis proposed in this paper takes the dependency of the successive decoding attempts into account. The proposed analysis is applicable for type-II hybrid ARQ schemes with known weight spectrum of the error-correcting codes used. The channel models considered include AWGN and Rayleigh fading. Numerical experiments show that the proposed analysis is accurate in capturing the performance of type-II hybrid ARQ systems at mid-to-high SNR region.**

I. Introduction

The concept of hybrid automatic-repeat-request (ARQ) is widely used for error control if a feedback channel is available. It combines the advantages of both forward-error-correction (FEC) and ARQ schemes so that it can provide lower error probability and higher throughput compared with FEC or ARQ scheme alone [1].

Conventionally, there are two categories of hybrid ARQ schemes, type-I and type-II hybrid ARQ schemes. Type-I hybrid ARQ is a straightforward combination of FEC and ARQ. In a nonstationary channel, type-II hybrid ARQ is usually prefered because it can provide adaptive error control capability. In a type-II hybrid ARQ scheme, the transmitter first encodes the information bits with an error detection code and transmits them either with or without redundant bits for error correction. If decoding error occurs at the receiver, the receiver asks for additional redundant bits from the transmitter and combines them with the previously received bits to form a more powerful codeword. The more the number of transmissions is, the longer and thus the more powerful the codeword would be. The retransmission procedure is repeatedly performed until a decoding success occurs at the receiver [2], [3]. However, if the channel condition is very bad, such schemes may result in a large delay. A more practical way is to limit the maximum number of retransmissions and the scheme is named as the truncated hybrid ARQ scheme [4]–[7].

As hybrid ARQ becomes more and more popular in modern wireless communications, it is desirable to have an accurate performance analysis of hybrid ARQ, in particular the more advanced type-II hybrid ARQ. In the literature, the error probability of type-II hybrid ARQ at stage i is usually approximated by the error probability of the combined codeword after i transmissions. The approximation results from the assumption that if the combined codeword is in error, decoding errors must also have been occurred in all of the previous $i-1$ transmissions. The assumption simplifies the analysis and is widely applied in the existing hybrid ARQ performance analysis [3], [5], [7]–[12]. However, the assumption is not generally true. Consider the case of transmitting a 2000-bit codeword in two transmissions. In the first transmission, 1500 bits of the codeword are sent, while the remaining 500 bits are sent in the second transmission. Due to the channel variation, it is possible to have very good channel condition in the first transmission and very bad channel condition in the second transmission. Given such channel realization occurs, the codeword can be successfully decoded from the 1500 bits received in the first transmission, while the combined codeword after the second transmission is erroneously decoded due to the severely damaged bits in the second transmission. As far as hybrid ARQ is concerned, the codeword should be considered as a success since it is already successfully decoded in the first transmission. However, in the aforementioned hybrid ARQ analysis papers, such case is counted as a decoding failure. This adds inaccuracy to the performance analysis in these papers. In some papers, such assumption is not used and the exact error probability is derived for type-II hybrid ARQ systems with Reed-Solomon-like codes [13], [14]. However, the results in these papers are not directly applicable to other commonly used codes like rate-compatible punctured convolutional codes (RCPC codes) [3], [5].

In this paper, we propose a performance analysis of type-II hybrid ARQ which takes the dependency of the successive decoding attempts into account. The channel models considered are additive white Gaussian noise (AWGN) and Rayleigh fading channels. The proposed type-II hybrid ARQ analysis is applicable to any error-correcting codes with known weight spectrum. Numerical experiments show that the proposed analysis is accurate in capturing the performance of type-II ARQ systems at mid-to-high SNR region. While the work proposed in this paper considers only binary phase shift keying (BPSK) modulation to simplify the analysis, the generalization of the result to bit-interleaved coded modulation (BICM) [15], [16] cases is included in our future works.

The remainder of this paper is organized as follows. In Section II, the system model of a general type-II hybrid ARQ scheme is described. The channel models considered are also given. In Section III, the error probability analysis is presented. In Section IV, numerical experiments are provided. Finally, the conclusions are addressed in Section V.

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This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2008 proceedings.

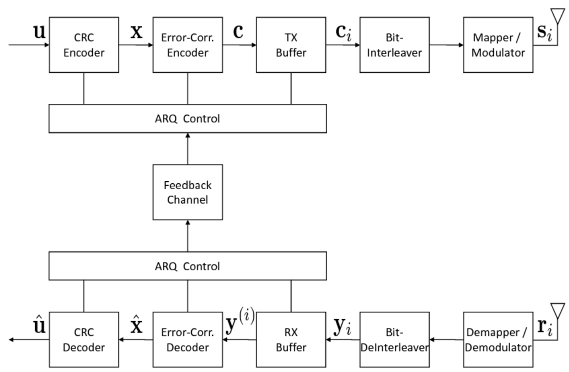

Fig. 1. Type-II hybrid ARQ system

II. System Model

The truncated type-II hybrid ARQ scheme considered is shown in Fig. 1. The transmitter consists of a cyclic redundancy check (CRC) encoder for error detection, an error-correction encoder, a transmitter (TX) buffer, an interleaver, and a modulator; the receiver consists of a demodulator, a deinterleaver, a receiver (RX) buffer, an error-correction decoder, and a CRC decoder. The modulation scheme considered in this work is BPSK.

At the transmitter side, the information sequence $\mathbf{u}$ is first encoded by the CRC encoder. The CRC encoded sequence $\mathbf{x}$ is then encoded by the error-correction encoder into a coded sequence $\mathbf{c} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N]$ which is stored at the TX buffer. The subsequence $\mathbf{c}_i = [c_{i,1}, c_{i,2}, \cdots c_{i,n_i}]$ denotes the coded information transmitted in the i-th transmission of the type-II hybrid ARQ scheme. Note that the maximum number of transmissions is denoted by N. The error-correction code is a general channel code with weight spectrum of the following form

$$A(D_1, D_2, \cdots, D_N) = \sum_{d_1, d_2, \cdots, d_N} a_{d_1, d_2, \cdots, d_N} D_1^{d_1} D_2^{d_2} \cdots D_N^{d_N}, \quad (1)$$

where $a_{d_1, d_2, \cdots, d_N}$ denotes the number of coded sequences $\mathbf{c}$ with Hamming weights

$$\begin{aligned} w(\mathbf{c}_1) &= d_1, \\ w(\mathbf{c}_2) &= d_2, \\ &\vdots \\ w(\mathbf{c}_N) &= d_N. \end{aligned} \quad (2)$$

Finally the coded subsequence $\mathbf{c}_i$ of the i-th transmission is interleaved by π_i and modulated into signal sequence

$$\mathbf{s}_i = \sqrt{E_s}[(-1)^{\tilde{c}_{i,1}}, (-1)^{\tilde{c}_{i,2}}, \cdots, (-1)^{\tilde{c}_{i,n}}] \quad (3)$$

for transmission, where $\tilde{\mathbf{c}}_i = [\tilde{c}_{i,1}, \tilde{c}_{i,2}, \cdots \tilde{c}_{i,n_i}] = \pi_i(\mathbf{c}_i)$.

At the receiver side, the sampled matched-filter output $\mathbf{r}_i = [r_{i,1}, r_{i,2}, \cdots, r_{i,n_i}]$ of the i-th transmission is given by

$$r_{i,k} = \alpha_{i,k}\sqrt{E_s}(-1)^{\tilde{c}_{i,k}} + n_{i,k}, \quad (4)$$

where $n_{i,k}$ is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$, i.e., $n_{i,k} \sim N(0, \frac{N_0}{2})$. The channel side information (CSI) is assumed to be known at the receiver and the scalar $\alpha_{i,k} \geq 0$ is used to denote the channel gain affecting the k-th symbol in the i-th transmission. In this paper, we assume that either $\alpha_{i,k} = 1$ resulted from an AWGN channel; or $\alpha_{i,k}^2 \sim \text{EXP}(1)$ due to a normalized Rayleigh fading channel. Moreover, we assume the channel interleaving is nearly perfect and thus each symbol experiences independent fading realization from the other symbols in the same transmission. The receiver deinterleaves $\mathbf{r}_i$ into $\mathbf{y}_i$ and stores it in the RX buffer. The error-correction decoder performs maximum likelihood (ML) decoding on the buffered sequence $\mathbf{y}^{(i)} = [\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_i]$. The decoded result $\hat{\mathbf{x}}$ is then checked by the CRC. If there is no error, the receiver sends an acknowledgement (ACK) to the transmitter to end the transmit process; otherwise the receiver sends a negative acknowledgement (NAK) to request for the coded subsequence $\mathbf{c}_{i+1}$ in the next transmission. If error still exists in $\hat{\mathbf{x}}$ after N transmissions, the receiver will give up and the transmit process is ended. In this work, the main focus is on the effectiveness of the error-correction code with the hybrid ARQ scheme, so an error-free feedback channel is assumed in this work to simplify the analysis. Due to the fact that the undetected error probability is very small with well-designed CRC codes, it is also assumed that the CRC check is perfect in detecting errors.

III. Error Probability Analysis

Let R_i denote the decoded result of the i-th transmission, $R_i = 1$ if $\mathbf{y}^{(i)}$ is decoded correctly, otherwise $R_i = 0$. Define E_i to be the event that the decoded results of the first i transmissions are all in error, thus

$$\begin{aligned} P\{E_i\} &= P\left\{\bigcap_{j=1}^{i}\{R_j = 0\}\right\} \\ &= P\left\{R_i = 0 \mid \bigcap_{j=1}^{i-1}\{R_j = 0\}\right\} P\left\{\bigcap_{j=1}^{i-1}\{R_j = 0\}\right\}. \end{aligned} \quad (5)$$

While event $R_i = 0$ has dependency on the events $\bigcap_{j=1}^{i-1}\{R_j = 0\}$, the decoded result R_i of the i-th transmission and $\{R_1, R_2, \cdots, R_{i-2}\}$ are nearly uncorrelated given the decoded result R_{i-1} of the previous transmission. Hence we have the approximation

$$P\left\{R_i = 0 \mid \bigcap_{j=1}^{i-1}\{R_j = 0\}\right\} \approx P\left\{R_i = 0 \mid R_{i-1} = 0\right\}. \quad (6)$$

Equation (5) can then be expanded as

$$\begin{aligned} & P\{E_i\} \\ \approx & \prod_{j=2}^{i} P\left\{R_j = 0 \;\middle|\; R_{j-1} = 0\right\} \times P\{R_1 = 0\} \\ = & \prod_{j=2}^{i} \left(1 - \frac{P\{R_j = 1 \cap R_{j-1} = 0\}}{P\{R_{j-1} = 0\}}\right) \times P\{R_1 = 0\}. \quad (7) \end{aligned}$$

Note that $P\{R_j = 1 \mid R_{j-1} = 0\}$ is used to replace $P\{R_j = 0 \mid R_{j-1} = 0\}$ which can simplify the analysis afterwards. The performance of the hybrid ARQ system is largely determined by $P\{E_i\}$. For instance, the probability of transmission failure is simply $P\{E_N\}$; the number of transmissions T needed until successful decoding has the probability distribution

$$P\{T = m\} = P\left\{R_m = 1 \;\middle|\; R_{m-1} = 0\right\} P\{E_{m-1}\}. \quad (8)$$

To compute $P\{E_i\}$ analytically from (7), one need to know $P\{R_{j-1} = 0\}$ and $P\{R_j = 1 \cap R_{j-1} = 0\}$, $j = 2, \cdots, i$. The former can be computed through the well known union bound

$$P\{R_j = 0\} = \sum_{d_1, \cdots, d_j} a_{d_1, \cdots, d_j} \cdot \; P_e\left(\sum_{k=1}^{j} d_k\right), \quad (9)$$

where $P_e(d)$ is the pairwise error probability of a codeword pair with Hamming distance d. By using the integral expression of the Q-function [17],

$$\mathrm{Q}(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{(-x^2/2\sin^2\theta)} d\theta, \quad (10)$$

one can derive

$$P_e(d) = \mathrm{Q}\left(\sqrt{\frac{2E_s d}{N_0}}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{E_s d}{N_0 \sin^2\theta}} \; d\theta \quad (11)$$

in AWGN, and

$$\begin{aligned} P_e(d) &= \mathrm{E}_T\left[\mathrm{Q}\left(\sqrt{\frac{2E_s T}{N_0}}\right)\right] \\ &= \frac{1}{\pi}\int_{\theta=0}^{\frac{\pi}{2}} \mathcal{M}_T\left(\frac{-E_s}{N_0 \sin^2\theta}\right) \; d\theta \quad (12) \end{aligned}$$

in Rayleigh fading case, where $T \sim \mathit{GAMMA}(d, 1)$ and $\mathcal{M}_T(s) = \mathrm{E}_T\left[e^{sT}\right] = \left(\frac{1}{1-s}\right)^d$ is the moment generating function of T. Now it is suffice to derive the exact expression of $P\{R_j = 1 \cap R_{j-1} = 0\}$ over AWGN channel and Rayleigh fading channel.

A. AWGN Channel

Assume that ML decoding is performed at the receiver. Event $\{R_i = 1 \cap R_{i-1} = 0\}$ indicates that the received signal $\mathbf{r}^{(i)} = [\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_i]$ is closer to the transmitted signal $\mathbf{s}^{(i)} = [\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_i]$ than to any other signal vector while $\mathbf{s}^{(i-1)}$ is not the nearest to $\mathbf{r}^{(i-1)}$. Define $B(\mathbf{x}, \mathbf{x}')$ to be the event that $\mathbf{r}^{(i-1)}$ is closer to signal vector $\mathbf{s}'^{(i-1)}$ (generated from $\mathbf{x}'$) than to $\mathbf{s}^{(i-1)}$ (generated from $\mathbf{x}$) but after the i-th transmission $\mathbf{r}^i$ becomes closer to signal vector $\mathbf{s}^i$ (generated from $\mathbf{x}$) than to $\mathbf{s}'^i$ (generated from $\mathbf{x}'$). With linear code used, we have

$$P\{R_i = 1 \cap R_{i-1} = 0\} \leq \sum_{\mathbf{x}' : \mathbf{x}' \neq \mathbf{0}} P\{B(\mathbf{0}, \mathbf{x}')\}. \quad (13)$$

The bound can be used as an close approximation of $P\{R_i = 1 \cap R_{i-1} = 0\}$ at mid-to-high SNR region. To compute $P\{B(\mathbf{0}, \mathbf{x}')\}$, we start from the definition of the event $B(\mathbf{x}, \mathbf{x}')$

$$\begin{aligned} & P\{B(\mathbf{x}, \mathbf{x}')\} \\ = & P\Big\{\|\mathbf{r}^{(i-1)} - \mathbf{s}^{(i-1)}\|^2 > \|\mathbf{r}^{(i-1)} - \mathbf{s}'^{(i-1)}\|^2 \\ & \cap \|\mathbf{r}^{(i)} - \mathbf{s}^{(i)}\|^2 < \|\mathbf{r}^{(i)} - \mathbf{s}'^{(i)}\|^2\Big\} \\ = & P\Big\{\mathbf{r}^{(i-1)} \cdot \left(\mathbf{s}'^{(i-1)} - \mathbf{s}^{(i-1)}\right) > 0 \\ & \cap \mathbf{r}_i \cdot (\mathbf{s}'_i - \mathbf{s}_i) + \mathbf{r}^{(i-1)} \cdot \left(\mathbf{s}'^{(i-1)} - \mathbf{s}^{(i-1)}\right) < 0\Big\}, \quad (14) \end{aligned}$$

where the last line of the equation is derived using $\|\mathbf{r}^{(i)} - \mathbf{s}^{(i)}\|^2 = \|\mathbf{r}^{(i-1)} - \mathbf{s}^{(i-1)}\|^2 + \|\mathbf{r}_i - \mathbf{s}_i\|^2$ and $\|\mathbf{r}^{(i)} - \mathbf{s}'^{(i)}\|^2 = \|\mathbf{r}^{(i-1)} - \mathbf{s}'^{(i-1)}\|^2 + \|\mathbf{r}_i - \mathbf{s}'_i\|^2$. Recall (3) and (4), we can obtain

$$\begin{aligned} & P\{B(\mathbf{0}, \mathbf{x}')\} \\ = & P\Bigg\{\sum_{\substack{j<i \\ c_{j,k} \neq c'_{j,k}}} n_{j,k} < -\sqrt{E_s}\sum_{j=1}^{i-1} d_j \\ & \cap \sum_{\substack{j<i \\ c_{j,k} \neq c'_{j,k}}} n_{j,k} + \sum_{c_{i,k} \neq c'_{i,k}} n_{i,k} > -\sqrt{E_s}\sum_{j=1}^{i} d_j\Bigg\} \\ = & P\left\{X_i < -W_i\sqrt{E_s} \cap X_i + Y_i > -\left(W_i + d_i\right)\sqrt{E_s}\right\} \\ = & \int_{x=-\infty}^{-W_i\sqrt{E_s}} f_{X_i}(x) \int_{y=-x-(W_i+d_i)\sqrt{E_s}}^{\infty} f_{Y_i}(y) dy dx \\ \triangleq & \Lambda_{\mathrm{A}}(W_i, d_i, \frac{E_s}{N_0}), \quad (15) \end{aligned}$$

where $X_i = \sum_{\substack{j<i \\ c_{j,k} \neq c'_{j,k}}} n_{j,k} \sim N\left(0, \frac{W_i \cdot N_0}{2}\right)$, $Y_i = \sum_{c_{i,k} \neq c'_{i,k}} n_{i,k} \sim N\left(0, \frac{d_i \cdot N_0}{2}\right)$, and $W_i = \sum_{j=1}^{i-1} d_j$. The expression of $\Lambda_{\mathrm{A}}(W_i, d_i, \frac{E_s}{N_0})$ is derived to be of the form in (16) using the integral expression of Q-function,

$$\Lambda_{\mathrm{A}}(W_i, d_i, \frac{E_s}{N_0}) = \int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \sum_{j=1}^{5} C_j \; \exp\left\{-\frac{E_s}{N_0} \cdot Z_j\right\} d\phi \; d\theta, \quad (16)$$

where $\{C_j\}$ and $\{Z_j\}$ are functions of (W_i, d_i, θ, ϕ) defined as

$$\begin{aligned}
C_1 &\triangleq \frac{2}{\pi^2}\sqrt{\frac{d_i \sin^2\theta}{d_i \sin^2\theta + W_i}},\\
C_2 &\triangleq \frac{2}{\pi^2},\ C_3 \triangleq -\frac{2}{\pi^2},\\
C_4 &\triangleq -\frac{2}{\pi^2}\sqrt{\frac{d_i \sin^2\theta}{d_i \sin^2\theta + W_i}},\\
C_5 &\triangleq \frac{1}{\pi^2}\sqrt{\frac{d_i \sin^2\theta}{d_i \sin^2\theta + W_i}},\\
Z_1 &\triangleq \frac{(W_i+d_i)^2\,(W_i \sin^2\phi + d_i \sin^2\theta)}{(d_i \sin^2\theta + W_i) W_i \sin^2\phi},\\
Z_2 &\triangleq \frac{W_i}{\sin^2\theta},\ Z_3 \triangleq \frac{(W_i+d_i)^2}{W_i \sin^2\theta},\\
Z_4 &\triangleq \frac{(W_i+d_i)^2}{d_i \sin^2\theta + W_i},\\
Z_5 &\triangleq \frac{(W_i+d_i)^2 \sin^2\theta \sin^2\phi + W_i d_i \cos^4\theta}{(d_i \sin^2\theta + W_i)\sin^2\theta \sin^2\phi}.
\end{aligned} \tag{17}$$

Define $a^{(i)}_{W_i,d_i} \triangleq \sum_{d_1+d_2+\cdots d_{i-1}=W_i} \sum_{d_{i+1}} \cdots \sum_{d_N} a_{d_1,d_2,\cdots,d_N}$, we can obtain the final form of $P\{R_i = 1 \cap R_{i-1} = 0\}$ as

$$P\{R_i = 1 \cap R_{i-1} = 0\} \lessapprox \sum_{W_i,d_i} a^{(i)}_{W_i,d_i} \Lambda_{\mathrm{A}}(W_i, d_i, \frac{E_s}{N_0}). \tag{18}$$

B. Rayleigh Fading Channel

For Rayleigh fading case, the conditional probability of $B(\mathbf{0}, \mathbf{x}')$ given the fading realization α is derived to be

$$P\left\{B(\mathbf{0}, \mathbf{x}')\middle|\alpha\right\} = \Lambda_{\mathrm{A}}(T_1, T_2, \frac{E_s}{N_0}), \tag{19}$$

where

$$\begin{aligned}
T_1 &= \sum_{\substack{j<i \\ c_{j,k}\neq c'_{j,k}}} \alpha^2_{j,k} \sim \mathit{GAMMA}(W_i, 1),\\
T_2 &= \sum_{c_{i,k}\neq c'_{i,k}} \alpha^2_{i,k} \sim \mathit{GAMMA}(d_i, 1).
\end{aligned} \tag{20}$$

To compute unconditional $P\{B(\mathbf{0}, \mathbf{x}')\}$, we need to compute $E_{T_1}[E_{T_2}[\Lambda_{\mathrm{A}}(T_1, T_2, \frac{E_s}{N_0})]]$. However it is very difficult to compute with the existence of T_2 here. To solve this problem, we introduce a probabilistic bounding trick here. Note that the larger T_2 is, the larger is $\Lambda_{\mathrm{A}}(T_1, T_2, \frac{E_s}{N_0})$. As a result, we try to substitute T_2 by a probabilistic upper bound kT_1 where k is chosen so that the probability of $\{T_2 > kT_1\}$ is less than a very small probability p. The probability of $\{T_2 > kT_1\}$ can be bounded as

$$\begin{aligned}
&P\{T_2 > kT_1\}\\
&= \int_{t_1=0}^{\infty} f_{T_1}(t_1)(1 - F_{T_2}(kt_1))dt_1\\
&= \sum_{i=0}^{d_i-1} \binom{i+W_i-1}{i} \frac{k^{i-1}}{(k+1)^{i+W_i-1}}\\
&\leq (k+1)^{-\min_i W_i} \sum_{i=0}^{d_i-1} \binom{i+W_i-1}{i},
\end{aligned} \tag{21}$$

where $f_{T_1}(t_1)$ is the probability density function of T_1, and $F_{T_2}(t_2)$ is the cumulative distribution function of T_2. Hence if k is chosen to be

$$k(p, W_i, d_i) \triangleq \left[\frac{1}{p}\sum_{i=0}^{d_i-1}\binom{i+W_i-1}{i}\right]^{\frac{1}{\min_i W_i}} - 1, \tag{22}$$

we can upper bound $P\{T_2 > kT_1\}$ by p. Substitute T_2 by $k(p, W_i, d_i)T_1$, we can derive $P\{B(\mathbf{0}, \mathbf{x}')\}$ as (23) on top of the next page, where $\mathcal{M}_{T_1}(s) = \mathrm{E}_{T_1}\left[e^{sT_1}\right]$ is the moment generating function of T_1, i.e.,

$$\mathcal{M}_{T_1}(s) = \left(\frac{1}{1-s}\right)^{W_i}. \tag{24}$$

The final form of $P\Big\{R_i = 1 \cap R_{i-1} = 0\Big\}$ can then be obtained as

$$P\Big\{R_i = 1 \cap R_{i-1} = 0\Big\} \lessapprox \sum_{W_i,d_i} a^{(i)}_{W_i,d_i} \Lambda_{\mathrm{R}}(W_i, d_i, \frac{E_s}{N_0}, p). \tag{25}$$

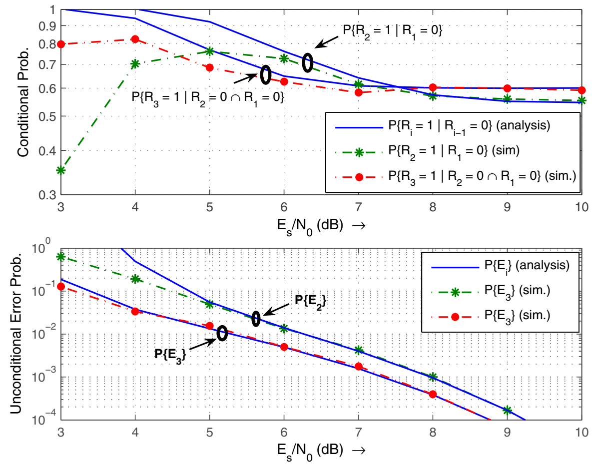

Fig. 2. Conditional probability $P\{R_i = 1 \mid R_{i-1} = 0\}$ and unconditional error probability $P\{E_i\}$ at stage $i = 1, 2$ of a type-II hybrid ARQ system over AWGN channel, $N = 3$ with code rates 8/9, 8/10 and 8/12.

IV. Numerical Results

Here we consider the performance of a truncated type-II hybrid ARQ system with maximum number of three transmissions. RCPC codes with decremental code rates 8/9, 8/10,

$$
\begin{aligned}
P\left\{B(\mathbf{0},\mathbf{x}')\right\} &= \mathrm{E}_{T_1}\left[\mathrm{E}_{T_2}\left[\Lambda_{\mathrm{A}}(T_1,T_2,\frac{E_s}{N_0})\right]\right] \\
&\lessapprox \mathrm{E}_{T_1}\left[\Lambda_{\mathrm{A}}(T_1,k(p,W_i,d_i)T_1,\frac{E_s}{N_0})\right] \\
&= \frac{2}{\pi^2}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}}\left\{\sqrt{\frac{k(p,W_i,d_i)\sin^2\theta}{k(p,W_i,d_i)\sin^2\theta+1}}\mathcal{M}_{T_1}\left(-\frac{E_s}{N_0}\frac{(1+k(p,W_i,d_i))^2\left(\sin^2\phi+k(p,W_i,d_i)\sin^2\theta\right)}{(k(p,W_i,d_i)\sin^2\theta+1)\sin^2\phi}\right)\right. \\
&\quad+\mathcal{M}_{T_1}\left(-\frac{E_s}{N_0}\frac{1}{\sin^2\theta}\right)-\mathcal{M}_{T_1}\left(-\frac{E_s}{N_0}\frac{(1+k(p,W_i,d_i))^2}{\sin^2\theta}\right) \\
&\quad-\sqrt{\frac{k(p,W_i,d_i)\sin^2\theta}{k(p,W_i,d_i)\sin^2\theta+1}}\mathcal{M}_{T_1}\left(-\frac{E_s}{N_0}\frac{(1+k(p,W_i,d_i))^2}{k(p,W_i,d_i)\sin^2\theta+1}\right) \\
&\quad\left.+\frac{1}{2}\sqrt{\frac{k(p,W_i,d_i)\sin^2\theta}{k(p,W_i,d_i)\sin^2\theta+1}}\mathcal{M}_{T_1}\left(-\frac{E_s}{N_0}\frac{(1+k(p,W_i,d_i))^2\sin^2\theta\sin^2\phi+k(p,W_i,d_i)\cos^4\theta}{(k(p,W_i,d_i)\sin^2\theta+1)\sin^2\theta\sin^2\phi}\right)\right\}d\phi\, d\theta \\
&\triangleq \Lambda_{\mathrm{R}}\left(W_i,d_i,\frac{E_s}{N_0},p\right)
\end{aligned}
\tag{23}
$$

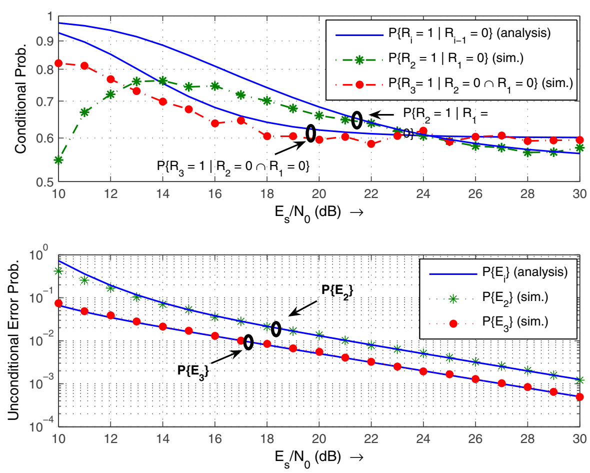

Fig. 3. Conditional probability $P\{R_i = 1 \mid R_{i-1} = 0\}$ and unconditional error probability $P\{E_i\}$ at stage $i = 1, 2$ of a type-II hybrid ARQ system over Rayleigh fading channel, $N = 3$ with code rates 8/9, 8/10 and 8/12.

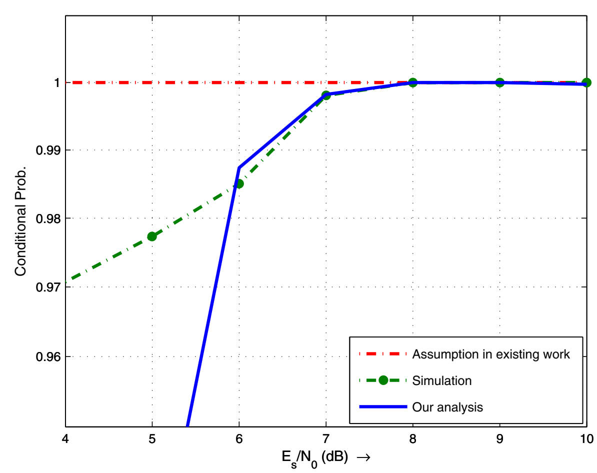

Fig. 4. Conditional probability $P\{R_{i-1} = 0 \mid R_i = 0\}$ over AWGN channel, $N = 2$ with code rates 8/9, and 8/10.

and 8/12 are used with the generators and puncturing matrices specified according to Table II(a) of [5]. The system performance under AWGN channel and Rayleigh fading are simulated and compared with analytical results computed from (18) and (25). In Figure 2 we demonstrate the conditional probability $P\{R_i = 1 \mid R_{i-1} = 0\}$ and the error probability $P\{E_i\}$ at stage $i = 2, 3$ under AWGN channel. Note that due to the high code rate, the coding gain is small, and stage 1 results are not shown here since it is simply the same as the non-ARQ case. We can see that the proposed analysis is accurate in capturing the conditional and unconditional error probabilities at each stage. In Figure 3, the error probabilities of the same type-II ARQ system under Rayleigh fading are evaluated using $p = 10^{-5}$ in Section III-B. It shows that our analysis is accurate in the fading case as well. Also, we compute the value of $E[T]$ by (8) and it is analytically 1.057 compared with simulation result 1.047 when SNR is 6dB under AWGN channel and 1.039 compared with 1.037 when SNR is 20dB under Rayleigh fading. In Figure 4, we plot the conditional probability $P\{R_1 = 0 \mid R_2 = 0\}$ under AWGN. As mentioned in Section I, most hybrid ARQ analysis work in the literature assumes $P\{R_1 = 0 \mid R_2 = 0\} = 1$. From the figure, we see that is not really the case in reality. On the other hand, the derived $P\{R_1 = 0 \mid R_2 = 0\}$ from our analysis is close to the simulation result. By taking the dependency of the successive decoding stages into account, it enables us to analyze the performance of type-II ARQ systems in more accurate manner.

V. Conclusion

In this paper, a performance analysis has been proposed for type-II hybrid ARQ systems in AWGN and Rayleigh fading environments. Unlike the existing work in the literature, the analysis takes the dependency of the successive decoding stages into consideration. Numerical results show that the proposed analysis is accurate in capturing the system performance at mid-to-high SNR regions. While this paper considers only BPSK modulation to simplify the analysis, the generalization of the result to bit-interleaved coded modulation (BICM) cases is included in our future works. The analysis gives us more insight to our follow-up work on the optimal design problem of type-II hybrid ARQ which is currently in progress.

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